

The comparison between the zeros of polynomials

Research Article

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Abstract:	The aim of this paper is to study of some bound theorems for the zeros of polynomials which are established in literature comparing between the zeros of polynomials by mean of examples
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1. Introduction

Locating the zeros of polynomials is a classical problem, which has attracted the attention of many mathematicians beginning with Cauchy. This problem, which is still a fascinating topic to both complex and numerical analysis, has many applications in diverse fields of mathematics. The Frobenius companion matrix plays an important link between matrix analysis and polynomials. It is used for the location for the zeros of polynomials by matrix method, It is also used for the numerical approximation.

Cauchy was the earliest contributors in the theory of the location of zeros of a polynomial, and since this subject has been studied by many interested others. Different upper and lower bounds for the moduli of the zeros are given in numerous publications. There is always a need for better and better results in this subject because of its application in many areas, including signal processing, communication theory, and control theory.

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Several mathematicians used matrix analysis methods to obtain new proofs of classical bounds for the zeros of polynomials and to derive new bounds for these zeros.

Definition 1.1

A monic polynomial is a single-variable polynomial (that is, a univariate polynomial) in which the leading coefficient (the nonzero coefficient of highest degree) is equal to 1.

Definition 1.2

Let $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$

be a monic polynomial of degree $n \geq 2$ with complex coefficients a_1, a_2, \dots, a_n , where $a_1 \neq 0$. Then the Frobenius companion matrix of p is given by

$$C(P) = \begin{bmatrix} -a_n & -a_{n-1} & \dots & -a_2 & -a_1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

Theorem 1.1 [1]

If z is any zero of $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, then Abdurakhmanov:

$$|Z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \right)$$

Theorem 1.2 [11]

If z is any zero of $p(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, then Kittaneh :

$$|Z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \right)$$

Theorem 1.3 [6]

If z is any zero of $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, then

Linden 1 :

$$z \leq \frac{|a_n|}{n} \left(\frac{n-1}{n} \left(n-1 + \sum_{j=1}^n |a_j|^2 - \frac{|a_n|^2}{n} \right) \right)^{\frac{1}{2}}$$

Theorem 1.4 [11]

If z is any zero of $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, then
BK (another bound for Kittaneh):

$$|z| \leq \frac{1}{2} \left(|a_n| + 1 + \sqrt{(|a_n| - 1)^2 + 4 \sqrt{\sum_{j=1}^{n-1} |a_j|^2}} \right).$$

Theorem 1.5 [4]

In Frobenius Companion Matrix.

If z is any zero of $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, then

$$\begin{aligned} 1. |z| &\leq \max\{|a_1|, 1 + |a_2|, 1 + |a_3|, \dots, 1 + |a_n|\} \\ &\leq 1 + \max\{|a_1|, |a_2|, |a_3|, \dots, |a_n|\} \end{aligned}$$

(Cauchy's bound)

$$\begin{aligned} 2. |z| &\leq \max\{1, |a_1| + |a_2| + |a_3| + \dots + |a_n|\} \\ &\leq 1 + |a_1| + |a_2| + |a_3| + \dots + |a_n| \end{aligned}$$

(Montel's bound)

$$3. |z| \leq (1 + |a_1|^2 + |a_2|^2 + |a_3|^2 + \dots + |a_n|^2)^{\frac{1}{2}}$$

(Carmichael-Mason's bound)

Theorem 1.6 [6]

If z is any zero of $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, then
Linden 2 :

$$|z| \leq \left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}$$

2 Main results

We present the relation between Abdurakhmanov bound and Kittaneh bound .

Theorem 2.1

Let $P(z) = z^n + a_n z^{n-1} + \dots + a_2 z + a_1$, $a_i \neq 0$ and z is any zero of p . Then
Abdurakhmanov bound $>$ kittaneh bound

Proof :

$$\begin{aligned} & \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \right) \\ & \geq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2} \right) \end{aligned}$$

If and only if,

$$\begin{aligned} & \frac{1}{2} |a_n| + \frac{1}{2} \cos \frac{\pi}{n} + \frac{1}{2} \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \\ & \geq \frac{1}{2} |a_n| + \frac{1}{2} \cos \frac{\pi}{n} + \frac{1}{2} \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2} \end{aligned}$$

If and only if,

$$\begin{aligned} & \frac{1}{2} \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \\ & \geq \frac{1}{2} \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2} \end{aligned}$$

If and only if,

$$\sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \geq \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2}$$

If and only if,

$$\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2 \geq \left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2$$

If and only if,

$$\left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2 \geq (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2$$

If and only if,

$$1 + 2\sqrt{\sum_{j=1}^{n-1} |a_j|^2} + \sum_{j=1}^{n-1} |a_j|^2 \geq |a_{n-1}|^2 + 2|a_{n-1}| + 1 + \sum_{j=1}^{n-2} |a_j|^2$$

If and only if,

$$2\sqrt{\sum_{j=1}^{n-1} |a_j|^2} \geq 2|a_{n-1}|$$

If and only if,

$$\sum_{j=1}^{n-1} |a_j|^2 \geq |a_{n-1}|^2$$

If and only if,

$$\sum_{j=1}^{n-2} |a_j|^2 \geq 0$$

and hence, we see that $\sum_{j=1}^{n-2} |a_j|^2 \geq 0$ always true .

So, the Kittaneh bound is better than Abdurakmanov because if Kittaneh isn't better than Abdurakmanov ,then $\sum_{j=1}^{n-2} |a_j|^2 \leq 0$, which is false.

Also, Kittaneh bound and Abdurakmanov bound aren't equal because if Kittaneh bound equal Abdurakmanov bound, then $\sum_{j=1}^{n-2} |a_j|^2 = 0$, and hence $|a_j|^2 = 0$, which is false because $a_1 \neq 0$.

In the following example, we show that Kittaneh bound is the better than Abdurakmanov bound .

Example 2.2

Let $p(z) = z^4 + 5$. Then

$$a_4 = 0, a_3 = 0, a_2 = 0, a_1 = 5$$

So,

$$\text{Abdurakmanov : } |z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \right) \leq 3.3743$$

Kittaneh :

$$|z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2} \right) \leq 2.9274$$

In the following example, we show that Kittaneh bound is the better than Abdurakmanov bound .

Example 2.3

Let $p(z) = z^4 + 3z^3 + 10z^2 + 2z + 1$. Then

$$a_4 = 3, a_3 = 10, a_2 = 2, a_1 = 1$$

So,

Abdurakmanov:

$$|z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \right) \leq 3.3743$$

Kittaneh :

$$|z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2} \right) \leq 2.9274$$

In the following example, we show that Kittaneh bound is the better than Abdurakmanov bound .

Example 2.4

Let $p(z) = z^6 + 3z^5 + 4z^4 + 2z^2 + 3$. Then

$$a_6 = 3, a_5 = 4, a_4 = 0, a_3 = 2, a_2 = 0, a_1 = 3$$

So,

Abdurakmanov :

$$|z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + \left(1 + \sqrt{\sum_{j=1}^{n-1} |a_j|^2} \right)^2} \right) \leq 16.97$$

Kittaneh :

$$|z| \leq \frac{1}{2} \left(|a_n| + \cos \frac{\pi}{n} + \sqrt{\left(|a_n| - \cos \frac{\pi}{n} \right)^2 + (|a_{n-1}| + 1)^2 + \sum_{j=1}^{n-2} |a_j|^2} \right) \leq 5.194$$

Now, we present the relation between Carmichael-Mason's bound and Linden's 2 bound

$$\left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2} \quad \text{with} \quad \sum_{j=1}^n |a_j|^2 > 1$$

Theorem 2.5

Let $p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \dots + a_2 z + a_1$

With $\sum_{j=1}^n |a_j|^2 > 1$. Then

$$\text{i)} \quad \left(1 + \sum_{j=1}^n |a_j|^2 \right)^{1/2} > \left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2}$$

if and only if, $1 > \sum_{j=2}^n |a_j|^2$

$$\text{ii)} \quad \left(1 + \sum_{j=1}^n |a_j|^2 \right)^{1/2} < \left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2}$$

if and only if, $1 < \sum_{j=2}^n |a_j|^2$

$$\text{iii)} \quad \left(1 + \sum_{j=1}^n |a_j|^2 \right)^{1/2} = \left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2}$$

if and only if, $1 = \sum_{j=2}^n |a_j|^2$

Proof :

$$\text{i) Since } \sum_{j=1}^n |a_j|^2 > 1 \quad ; \text{ we have } \max \left[1, \sum_{j=1}^n |a_j|^2 \right] = \sum_{j=1}^n |a_j|^2 .$$

Now ,

$$\left(1 + \sum_{j=1}^n |a_j|^2 \right)^{1/2} > \left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2} .$$

$$\text{If and only if, } \left(1 + \sum_{j=1}^n |a_j|^2 \right) > \sum_{j=1}^n |a_j|^2 + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2}$$

$$\text{if and only if } 1 > \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \text{ f and only if, } 1 > \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2}.$$

If we replace the greater than in the proof of (i) by less than we get (ii), and if we replace it by equality we get (iii) .

Example 2.6

i) For the case that bound $\left(1 + \sum_{j=1}^n |a_j|^2 \right)^{1/2}$ is greater than the bound

$$\left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2} \quad (a)$$

Consider $p(z) = z^3 + \frac{1}{2}z^2 + \frac{1}{2}z + 1$; in which Carmichael-Mason's bound =1.58

$$\left[\max \left(1, \sum_{j=1}^n |a_j|^2 \right) + \left(\sum_{j=2}^n |a_j|^2 \right)^{1/2} \right]^{1/2} = 1.485$$

Which implies that Carmichael-Mason's bound is greater than Linden's bound in (a) in this example.

ii) For the case Carmichael-Mason's bounds is less than Linden's bound in (a). Consider $p(z) = z^2 + 2z + 3$, in which Carmichael-Mason's bound = 3.605 and bound in (a) is equal to 3.87 , which implis that Carmichael-Mason's bound is less than Linden's bound in (a) in this example.

iii) For the case that Carmichael-Mason's bound and Linden's bound in (a) are equal. Consider $p(z) = z^2 + z + 4$, in which Carmichael-Mason's bound = Linden's bound in (a) = 4.

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